MATHEMATICALLY MODELLING THREE-DIMENSIONAL PLANT GROWTH FOR USE IN ADDITIVE MANUFACTURE

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Plant Growth, a Biological Analogue

Design for Additive Manufacture

- Additive manufacture (AM) is a bottom-up manufacturing approach that constructs parts using computer-aided design (CAD).
- AM gives rise to a greater design freedom compared to traditional subtractive methods (or "top-down" approaches).
- CAD adopts a *subtractive manufacturing approach*.
  - ⇒ To exploit the greater design freedom AM has to offer, the designer is required to think more creatively - **Design for Additive Manufacture** (**DfAM**).
- DfAM remains a fragmented, challenging process, but there are tools available to ease, or even overcome, these difficulties.

A Mathematical Approach

- A current tool available is **topological optimisation**.
- With a *subtractive mindset*, it introduces 'holes' into the design space, generating organic-looking structures that resemble structures in nature.
- Gives rise to '**bio-inspired design**' and, thus, my research proposal: employing mathematical modelling techniques to revolutionise DfAM.
  - ⇒ Expanding the creative scope of design and inspiring new innovative ideas by visualising AM as a **process of growth**: a bottom-up process.

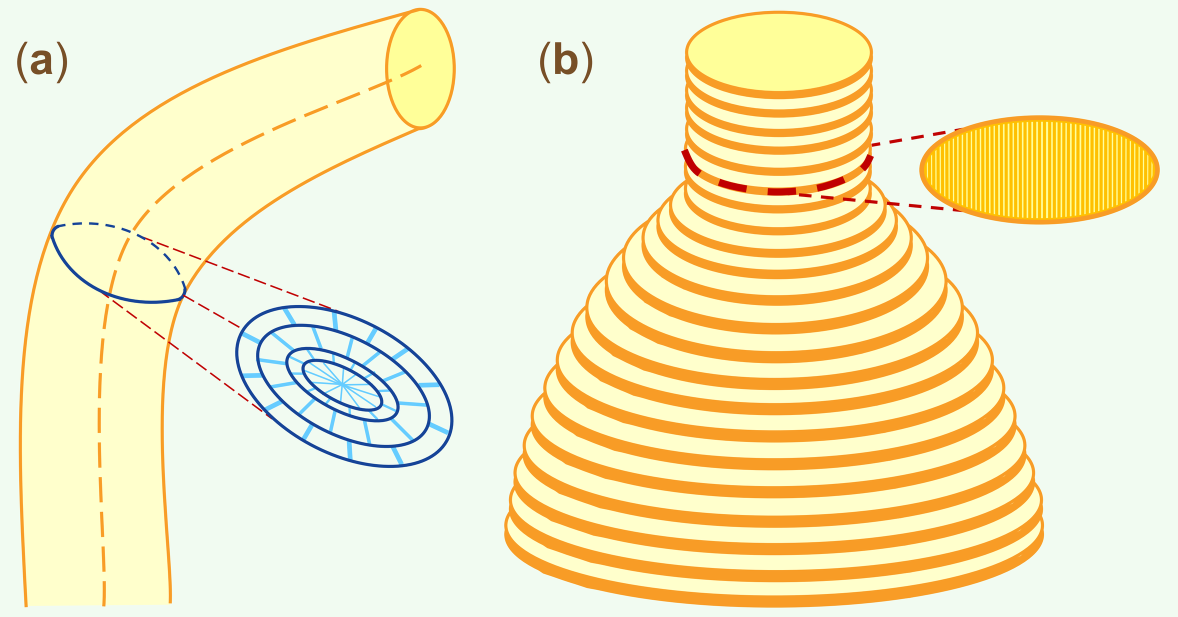


Figure 1: Two analogous processes. (a) presents a plant organ whose cross-section consists of layers of adhered cells and (b) presents the adhered layers of lines of extruded material that collectively form an arbitrary part.

An Analogous Process

- Despite both processes in Figure 1 being **three-dimensional**, the capabilities and constraints of plant growth are mirrored in AM.
- Just as plant cells are strongly adhered to one another, each layer of a part (the 'cell') depends on the one preceding it.
  - ⇒ In both cases, **cellular/material level deformation generates deformation at the tissue/part level.**
- Drawing upon these analogies will generate a technique to **visualise the dynamic evolution of a desired part** through a more familiar bottom-up approach.

Underlying Mathematical Modelling Techniques

From the Cellular Level...

Assuming each cell is subject to a turgor pressure P, we can employ the **Lockhart equation**[1] to determine the **relative elongation rate** (**RER**) of each cell wall segment in the cross-section Σ[2],

$$RER_{cell} = \begin{cases} \phi(T-Y), & T \geq Y, \\ 0, & T < Y, \end{cases}$$

with extensibility $\phi(\mathbf{x})$, axial tension $T(\mathbf{x})$, and yield $Y(\mathbf{x})$.

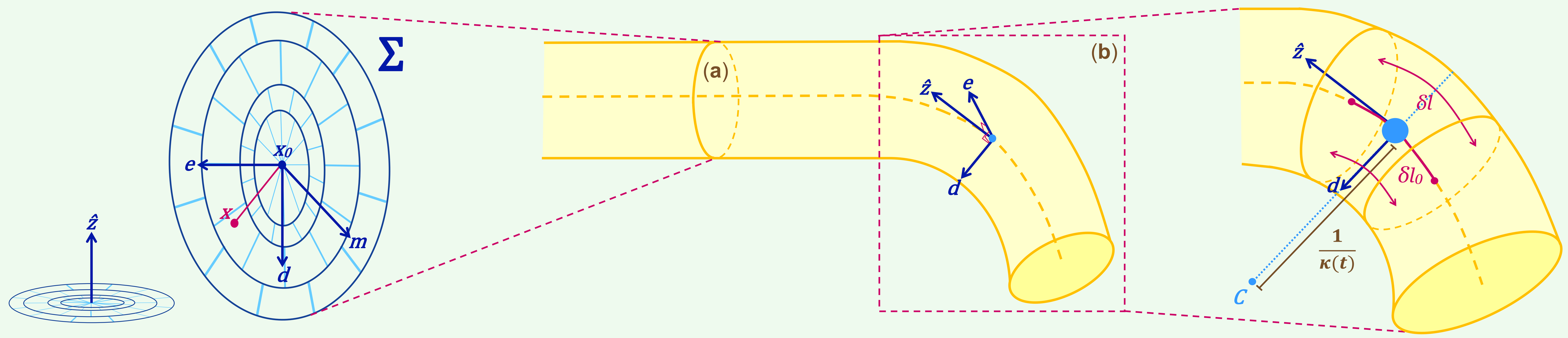


Figure 2: A magnification of (a) presents a cross-section Σ of N cells, with $\hat{\mathbf{z}}$ the tangent of the organ centre-line (pointing away from the tip), $\mathbf{d}$ the normal, $\mathbf{e}$ the binormal, $\mathbf{x_0}$ the centroid, $\mathbf{m}$ the moment, and $\mathbf{x}$ an arbitrary point. A magnification of (b) presents a segment of length δl_0 between two perpendicular cross-sections, with $\delta l > \delta l_0$. A point (blue) on the centre-line has radius of curvature $1/\kappa(t)$ and centre of curvature C, with $\kappa(t)$ the curvature of the centre-line at time t.

...to the Tissue Level

- An antisymmetric elongation of cells induces bending, producing an axial length[2] of,

$$\delta l(\mathbf{x},t) = \left(1+\kappa(t)f(\mathbf{x})\right)\delta l_0(t),$$

through $\mathbf{x}$, as seen in Figure 2, with $f(\mathbf{x}) = \mathbf{e}\cdot(\mathbf{x}-\mathbf{x_0})$.

- Assuming $|\kappa f| \ll 1$, generates the **tissue-level $RER(t)$ from RER_{cell}**[2],

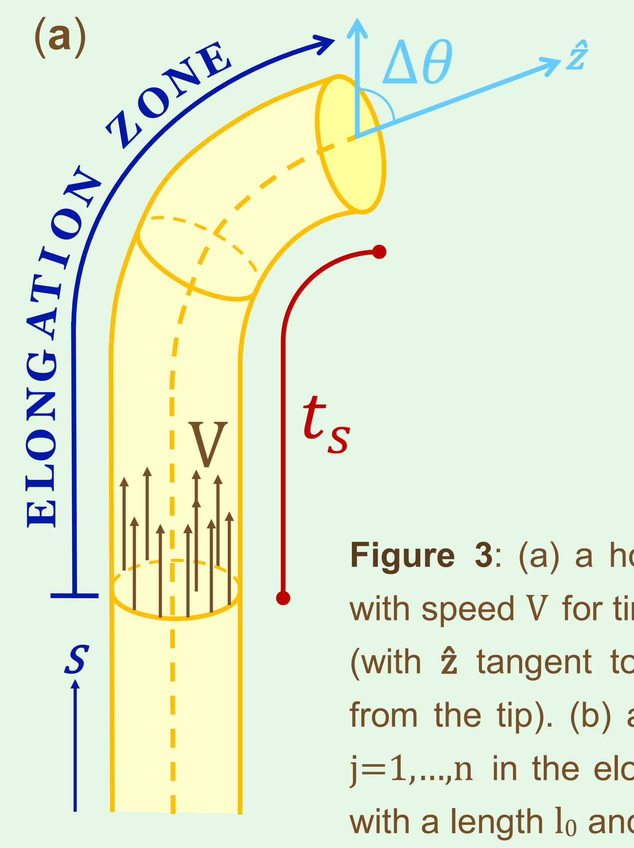
$$RER_{cell}(\mathbf{x},t) = \frac{1}{\delta l}\frac{d(\delta l)}{dt} \approx RER + \frac{d\kappa}{dt}f.$$

A Future for Bio-Inspired Design

The Bending Angle, $\Delta\theta$

- Bending of an organ is initiated by a **localised hormone signal**.
- Suppose cells in the elongation zone (EZ) experience a signal of magnitude A_0 for t_s time, travelling shootward with uniform speed $V = \beta l_0/c$, $\beta \gg 1$ (as seen in Figure 3).
- The length of cells exposed to the signal is Vt_s, and the **bending angle**[2] is calculated to be,

$$\Delta\theta = A_0\left(\frac{l_0 t_s}{c}\right)\beta\log_e\beta = A_0 V t_s \log_e\beta.$$



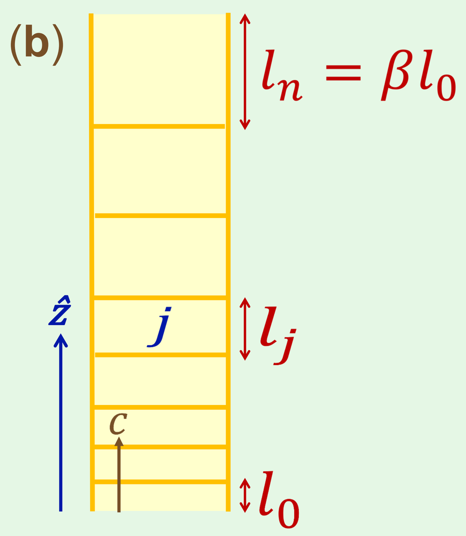


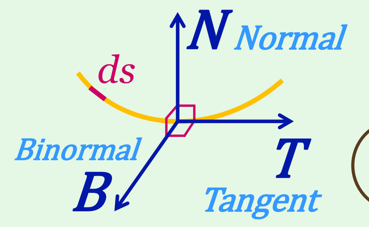
Figure 3: (a) a hormone signal travelling shootward S with speed V for time t_s, inducing bending $\Delta\theta$ of the root (with $\hat{\mathbf{z}}$ tangent to the root centre-line, pointing away from the tip). (b) a one-dimensional file of n root cells j=1,...,n in the elongation zone. Cells enter at a rate c with a length l_0 and leave with a length βl_0, $\beta \gg 1$.

Potential Explorations

① Warpage

Simulating **residual stress** within plant organs to understand when **warpage** may occur within a 3D printed component.

② Torsion



Employing the **Frenet-Serret formulae**,

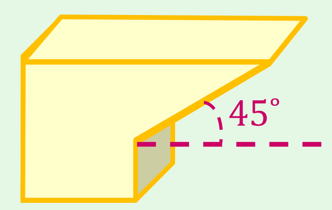
$$\begin{pmatrix} \frac{d\mathbf{T}}{ds} \\ \frac{d\mathbf{N}}{ds} \\ \frac{d\mathbf{B}}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix},$$

to introduce **torsion** τ into the model above.

③ Bending Angle

Utilising expressions for **bending angle** to:

- Optimise **the curvature** obtained by 3D printed components.
- Determine any **critical points** where supports are needed.



References

[1] Lockhart J.A. An Analysis of Irreversible Plant Cell Elongation. *Journal of Theoretical Biology*, **8**:264-275, 1965.
[2] Dyson R.J. et al. Mechanical Modelling Quantifies the Functional Importance of Outer Tissue Layers During Root Elongation and Bending. *New Phytologist*, **202**:1212-1222, 2014.